I B.Sc MATHS 15/09/2020 (1)

PAPER II – TRIGONOMETRY

UNIT - I

Expansions of $\cos n\theta$, $\sin n\theta$, Expansion of $\tan n\theta$ in terms of $\tan\theta$ – Expansion of $\tan(A+B+C+\ldots)$. Formation of Equations. [chapter III – section 1 to 3].

TEXT BOOK:

S. Narayan and T.K. Manickavachagom Pillay (2004) – Trigonometry – S. Viswanathan Printers & Publishers Pvt. Ltd. Chennai

Expansions for $\cos n\theta$ and $\sin n\theta$: (n being a positive integer).

De'moivre's theorem is

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \longrightarrow (1)$$

Binomial theorem is

$$(x+a)^n = x^n + nC_1 x^{n-1} a + nC_2 x^{n-2} a^2 + \ldots + nC_r x^{n-r} a^r + \ldots + nC_n a^n.$$

$nC_0 = 1$, $nC_1 = n$, $nC_2 = \frac{n(n-1)}{2}$, ..., $nC_{n-1} = n$, $nC_n = 1$

$$\therefore (\cos\theta + i\sin\theta)^n = \cos^n\theta + nC_1 \cos^{n-1}\theta\, i\sin\theta + nC_2 \cos^{n-2}\theta\,(i\sin\theta)^2 + nC_3 \cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots + nC_r \cos^{n-r}\theta\,(i\sin\theta)^r + \ldots + (i\sin\theta)^n \longrightarrow (2)$$

$$= \left[\cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta - \ldots\right] + i\left[nC_1\cos^{n-1}\theta\sin\theta - nC_3\cos^{n-3}\theta\sin^3\theta + nC_5\cos^{n-5}\theta\sin^5\theta - \ldots\right] \longrightarrow (3)$$

Equating the real and imaginary parts in (1) and (3)

we get

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta - \ldots$$

$$\sin n\theta = nC_1\cos^{n-1}\theta\sin\theta - nC_3\cos^{n-3}\theta\sin^3\theta + nC_5\cos^{n-5}\theta\sin^5\theta - \ldots$$

Note:

(1) The terms in the expansion of $\sin n\theta$ and $\cos n\theta$ are alternatively positive and negative.

(2) There are only finite number of terms in the expansions of $\cos n\theta$ and $\sin n\theta$.

(3) Both the series are in descending powers of $\cos\theta$ and in ascending powers of $\sin\theta$.

Problems.

(1) Expand

(a) $\cos 3\theta$ in terms of $\cos\theta$.

Solution: We know that

$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta + \cdots$

$\therefore \cos 3\theta = \cos^3\theta - 3C_2 \cos^{3-2}\theta \sin^2\theta.$

$= \cos^3\theta - 3\cos\theta\sin^2\theta.$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta.$

$= 4\cos^3\theta - 3\cos\theta.$

$3C_0 = 1$
$3C_1 = 3$
$3C_2 = \frac{3\times 2}{2} = 3$

(b) $\cos 4\theta$ purely in $\cos\theta$ and purely in $\sin\theta$.

Solution:

(i) purely in $\cos\theta$.

$\cos 4\theta = \cos^4\theta - 4C_2 \cos^2\theta\sin^2\theta + 4C_4 \sin^4\theta.$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2.$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + (1 - 2\cos^2\theta + \cos^4\theta).$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta.$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta.$

$= 1 - 8\cos^2\theta + 8\cos^4\theta.$

(or) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$

$4C_2 = \frac{4\times 3}{2} = 2\times 3 = 6$
$4C_4 = 1$

(ii) purely in $\sin\theta$.

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$
$$= (1-\sin^2\theta)^2 - 6(1-\sin^2\theta)\sin^2\theta + \sin^4\theta.$$
$$= 1 - 2\sin^2\theta + \sin^4\theta - (6 - 6\sin^2\theta)\sin^2\theta + \sin^4\theta.$$
$$= 1 - 2\sin^2\theta + \sin^4\theta - 6\sin^2\theta + 6\sin^4\theta + \sin^4\theta.$$
$$= 1 - 8\sin^2\theta + 8\sin^4\theta.$$

(c) $\sin 5\theta$ purely in $\sin\theta$.

Solution:

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \cdots$$
$$\therefore \sin 5\theta = {}^5C_1\cos^4\theta\sin\theta - {}^5C_3\cos^2\theta\sin^3\theta + {}^5C_5\cos^{5-5}\theta\sin^5\theta.$$
$$= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$$
$$= 5\cos^4\theta\sin\theta - 10\sin^3\theta(1-\sin^2\theta) + \sin^5\theta.$$

$${}^5C_3 = \frac{5\times4\times3}{1\times2\times3} = 10$$

$$= 5\sin\theta(1-\sin^2\theta)^2 - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta.$$
$$= 5\sin\theta(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta.$$
$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta.$$
$$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.$$

(d) $\cos 6\theta$ in terms of $\sin\theta$.

Solution:

$$\cos 6\theta = \cos^6\theta - {}^6C_2\cos^4\theta\sin^2\theta + {}^6C_4\cos^2\theta\sin^4\theta - {}^6C_6\sin^6\theta.$$
$$= (1-\sin^2\theta)^3 - 15(1-\sin^2\theta)^2\sin^2\theta + 15(1-\sin^2\theta)\sin^4\theta - \sin^6\theta.$$
$$= 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta - 15(1 - 2\sin^2\theta + \sin^4\theta)\sin^2\theta + 15(\sin^4\theta - \sin^6\theta) - \sin^6\theta.$$
$$= 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta - 15\sin^2\theta + 30\sin^4\theta - 15\sin^6\theta + 15\sin^4\theta - 15\sin^6\theta - \sin^6\theta.$$
$$= 1 - 18\sin^2\theta + 48\sin^4\theta - 32\sin^6\theta.$$

(1) (e) $\frac{\sin 3\theta}{\sin\theta}$ in terms of $\cos\theta$.

Solution:

$$\sin 3\theta = 3C_1\cos^2\theta\sin\theta - 3C_3\sin^3\theta.$$
$$= 3\cos^2\theta\sin\theta - \sin^3\theta.$$
$$\therefore \frac{\sin 3\theta}{\sin\theta} = 3\cos^2\theta - \sin^2\theta.$$
$$= 3\cos^2\theta - (1-\cos^2\theta).$$
$$= 3\cos^2\theta - 1 + \cos^2\theta.$$
$$= 4\cos^2\theta - 1.$$

(2) Prove the following results.

(a) $\frac{\sin 7\theta}{\sin\theta} = 64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1.$

Solution:

By Demoivre's theorem,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$
$$(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta. \longrightarrow (1)$$
$$\therefore (\cos\theta + i\sin\theta)^7 = \cos^7\theta + 7C_1\cos^6\theta(i\sin\theta) + 7C_2\cos^5\theta(i\sin\theta)^2 + 7C_3\cos^4\theta(i\sin\theta)^3 + 7C_4\cos^3\theta(i\sin\theta)^4 + 7C_5\cos^2\theta(i\sin\theta)^5 + 7C_6\cos\theta(i\sin\theta)^6 + 7C_7(i\sin\theta)^7.$$
$$= \left[\cos^7\theta - 7C_2\cos^5\theta\sin^2\theta + 7C_4\cos^3\theta\sin^4\theta - 7C_6\cos\theta\sin^6\theta\right] + i\left[7C_1\cos^6\theta\sin\theta - 7C_3\cos^4\theta\sin^3\theta + 7C_5\cos^2\theta\sin^5\theta - \sin^7\theta\right]. \longrightarrow (2)$$

Equating imaginary parts in (1) and (2), we get

$$\sin 7\theta = 7C_1\cos^6\theta\sin\theta - 7C_3\cos^4\theta\sin^3\theta + 7C_5\cos^2\theta\sin^5\theta - \sin^7\theta.$$

Dividing by $\sin\theta$, we get

$$\frac{\sin 7\theta}{\sin\theta} = 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta.$$

$$= 7\cos^6\theta - 35\cos^4\theta(1-\cos^2\theta) + 21\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$

$$= 7\cos^6\theta - 35\cos^4\theta + 35\cos^6\theta + 21\cos^2\theta(1-2\cos^2\theta+\cos^4\theta) - (1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta).$$

$$= 7\cos^6\theta - 35\cos^4\theta + 35\cos^6\theta + 21\cos^2\theta - 42\cos^4\theta + 21\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta.$$

$$= 64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1.$$

(2) (6) $\cos 8\theta = 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta.$

Solution:

By Demoivre's theorem

$$(\cos\theta + i\sin\theta)^8 = \cos 8\theta + i\sin 8\theta \longrightarrow (1)$$

By binomial theorem,

$$(\cos\theta + i\sin\theta)^8 = \cos^8\theta + 8C_1\cos^7\theta(i\sin\theta) + 8C_2\cos^6\theta(i\sin\theta)^2 + 8C_3\cos^5\theta(i\sin\theta)^3 + 8C_4\cos^4\theta(i\sin\theta)^4 + 8C_5\cos^3\theta(i\sin\theta)^5 + 8C_6\cos^2\theta(i\sin\theta)^6 + 8C_7\cos\theta(i\sin\theta)^7 + 8C_8(i\sin\theta)^8.$$

$$= [\cos^8\theta - 8C_2\cos^6\theta\sin^2\theta + 8C_4\cos^4\theta\sin^4\theta - 8C_6\cos^2\theta\sin^6\theta + 8C_8\sin^8\theta]$$

$$+ i[8C_1\cos^7\theta\sin\theta - 8C_3\cos^5\theta\sin^3\theta + 8C_5\cos^3\theta\sin^5\theta - 8C_7\cos\theta\sin^7\theta] \longrightarrow (2)$$

Equating real parts in (1) and (2), we get

$\cos 8\theta = \cos^8\theta - 8C_2 \cos^6\theta \sin^2\theta + 8C_4 \cos^4\theta \sin^4\theta$
$\qquad - 8C_6 \cos^2\theta \sin^6\theta + 8C_8 \sin^8\theta.$

$= (1-\sin^2\theta)^4 - 28\sin^2\theta(1-\sin^2\theta)^3 + 70\sin^4\theta(1-\sin^2\theta)^2$
$\qquad - 28\sin^6\theta(1-\sin^2\theta) + \sin^8\theta.$

$= (1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta) -$
$28\sin^2\theta(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta) -$
$70\sin^4\theta(1 - 2\sin^2\theta + \sin^4\theta) - 28\sin^6\theta(1-\sin^2\theta) + \sin^8\theta.$

$= 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta - 28\sin^2\theta$
$- 54\sin^4\theta + 54\sin^6\theta - 28\sin^8\theta - 70\sin^4\theta$
$- 140\sin^6\theta + 70\sin^8\theta - 28\sin^6\theta + 28\sin^8\theta + \sin^8\theta.$

$= 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 120\sin^8\theta.$

(2) (c) $\dfrac{\sin 5\theta}{\sin\theta} = 5 - 20\sin^2\theta + 16\sin^4\theta.$

$\qquad = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$

(d) $\cos 6\theta = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta.$ HW · Narayan Pg. 64 Q 2

(e) $\dfrac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$

$\qquad = 8\sin^4\theta - 8\sin^2\theta + 1.$

(f) $\cos 4\theta = 8\sin^4\theta - 8\sin^2\theta + 1.$ $35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$

(g) $\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$

$\sin 7\theta$ … $= \cos^7\theta \ldots$ $21\cos^2\theta\sin^5\theta - \sin^7\theta.$

(h) $\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta.$

Hence find $\dfrac{\sin 7\theta}{\sin\theta}$ in terms of $\sin\theta$.

Ans $7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$

(i) $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

(j) $\cos 9\theta = 256\cos^9\theta - 576\cos^7\theta + 432\cos^5\theta - 120\cos^3\theta$
$\qquad + 9\cos\theta.$

(k) $\dfrac{\sin 8\theta}{\sin\theta} = 128\cos^7\theta - 192\cos^5\theta + 80\cos^3\theta - 8\cos\theta$

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Expansion of $\tan n\theta$.

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}.$$

$$= \frac{nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + \cdots}{\cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta - \cdots}.$$

Dividing numerator and denominator by $\cos^n\theta$, we get

$$\boxed{\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + nC_5 \tan^5\theta - \cdots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta - \cdots}}$$

Problems

3. (a) Expand $\tan 7\theta$ in terms of $\tan\theta$

Solution:

We know that $\tan n\theta = \dfrac{nC_1 \tan\theta - nC_3 \tan^3\theta + nC_5 \tan^5\theta - \cdots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta - \cdots}$

$$\therefore \tan 7\theta = \frac{7C_1 \tan\theta - 7C_3 \tan^3\theta + 7C_5 \tan^5\theta - 7C_7 \tan^7\theta}{1 - 7C_2 \tan^2\theta + 7C_4 \tan^4\theta - 7C_6 \tan^6\theta}.$$

$$= \frac{7\tan\theta - 35\tan^3\theta + 21\sin^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}.$$

$7C_1 = 7$

$7C_2 = \frac{7\times 6}{1\times 2} = 21$

$7C_3 = \frac{7\times 6\times 5}{1\times 2\times 3} = 35$

$7C_4 = 35$

$7C_5 = 21$

$7C_6 = 7$

$7C_7 = 1$

3 (b) Expand $\tan 6\theta$ in powers of $\tan\theta$.

Solution:

$$\tan 6\theta = \frac{6C_1 \tan\theta - 6C_3 \tan^3\theta + 6C_5 \tan^5\theta}{1 - 6C_2 \tan^2\theta + 6C_4 \tan^4\theta - 6C_6 \tan^6\theta}$$

$$= \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}.$$

$6C_2 = \frac{6\times 5}{1\times 2} = 15$

$6C_3 = \frac{6\times 5\times 4}{1\times 2\times 3} = 20$

$6C_4 = 15$

$6C_5 = 6$

$6C_6 = 1$

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3.3 Expansion of $\tan(A+B+C+\cdots)$

We know that

~~cos (A+B+C~~

$(\cos A + i\sin A)(\cos B + i\sin B)(\cos C + i\sin C)(\cos D + i\sin D)\cdots$

$= \cos(A+B+C+\cdots) + i\sin(A+B+C+\cdots)$

i.e. $\cos(A+B+C+\cdots) + i\sin(A+B+C+\cdots)$

$= (\cos A + i\sin A)(\cos B + i\sin B)(\cos C + i\sin C)\cdots$

$= \cos A(1+i\tan A)\cos B(1+i\tan B)\cos C(1+i\tan C)\cdots$

$= [\cos A\cos B\cos C\cdots][(1+i\tan A)(1+i\tan B)(1+i\tan C)\cdots]$

$= \cos A\cos B\cos C\cdots[1 + i(\tan A + \tan B + \tan C + \cdots) + i^2(\tan A \tan B + \tan B \tan C + \cdots) + i^3(\tan A \tan B \tan C + \cdots) + \cdots]$

$= \cos A\cos B\cos C\cdots[1 + iS_1 + i^2S_2 + i^3S_3 + \cdots]$.

Equating real parts and imaginary parts, we get

$\cos(A+B+C+\cdots) = \cos A\cos B\cos C\cdots(1 - S_2 + S_4 - \cdots) \rightarrow (1)$

& $\sin(A+B+C+\cdots) = \cos A\cos B\cos C(S_1 - S_3 + S_5 - \cdots) \rightarrow (2)$

Dividing (2) by (1), we get

$$\boxed{\tan(A+B+C+\cdots) = \frac{S_1 - S_3 + S_5 - \cdots}{1 - S_2 + S_4 - \cdots}}$$

where $S_1 = \tan A + \tan B + \tan C + \cdots$

$S_2 = \tan A\tan B + \tan A\tan C + \tan B\tan C + \cdots$

$S_3 = \tan A\tan B\tan C + \tan A\tan B\tan D + \cdots$

$\cdots\cdots$

Note:

If $A = B = C \cdots = \theta$,

then $S_1 = nC_1 \tan\theta$.

$S_2 = nC_2 \tan^2\theta$

$S_3 = nC_3 \tan^3\theta \cdots$

$$\Rightarrow \tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + nC_5 \tan^5\theta - \cdots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta - \cdots}$$

Problems

(4) (a) If α, β, γ are the roots of the equation $x^3 + px^2 + qx + p = 0$, $q \neq 0$, prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians except in one particular case.

Eg 3 Narayan

Solution: Given equation is $x^3 + px^2 + qx + p = 0 \longrightarrow (1)$

Also α, β, γ are roots of (1).

$$\therefore \alpha + \beta + \gamma = -\frac{\text{Coeff. of } x^2}{\text{Coeff. of } x^3} = -p \quad \left[\text{Sum of the roots}\right]$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{\text{Coeff. of } x}{\text{Coeff. of } x^3} = q$$

$$\alpha\beta\gamma = -\frac{\text{Constant}}{\text{Coeff of } x^3} = -p.$$

Now if $A = \tan^{-1}\alpha$, $B = \tan^{-1}\beta$, $C = \tan^{-1}\gamma$,

then $\tan(A+B+C) = \dfrac{S_1 - S_3}{1 - S_2} = \dfrac{-p+p}{1-q} = 0$ if $q \neq 1$.

(ie) $\tan(A+B+C) = 0 = \tan n\pi$.

$\Rightarrow A + B + C = n\pi$.

$\Rightarrow \boxed{\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi}$ radians except in one case $q \neq 1$.

(3) (c) If $t=\tan\theta$, prove the following results: (10)

(i) $\tan 4\theta = \dfrac{4t-4t^3}{1-6t^2+t^4}$.

(ii) $\tan 5\theta = \dfrac{5t-10t^3+t^5}{1-10t^2+5t^4}$. (iii) $\tan 9\theta = ?$

(4) (b) Prove that the equation $\sin 3\theta = a\sin\theta + b\cos\theta + c$ has six roots and that the sum of the six values of θ which satisfy it is equal to an odd multiple of π radians.

Solution: Given $\sin 3\theta = a\sin\theta + b\cos\theta + c$.

$\Rightarrow 3\sin\theta - 4\sin^3\theta = a\sin\theta + b\cos\theta + c$,

Taking $t=\tan\theta/2 \Rightarrow 3\left(\dfrac{2t}{1+t^2}\right) - 4\left(\dfrac{2t}{1+t^2}\right)^3 = a\left(\dfrac{2t}{1+t^2}\right) + b\left(\dfrac{1-t^2}{1+t^2}\right) + c.$

$\Rightarrow \dfrac{6t}{1+t^2} - 4\cdot\dfrac{8t^3}{(1+t^2)^3} = \dfrac{2at}{1+t^2} + \dfrac{b-bt^2}{1+t^2} + c.$

$\Rightarrow 6t(1+t^2)^2 - 2at(1+t^2)^2 - (b-bt^2)(1+t^2)^2 - 32t^3 - c(1+t^2)^3 = 0.$

$\Rightarrow 6t(1+t^4+2t^2) - 2at(1+t^4+2t^2) - (b-bt^2)(1+t^4+2t^2) - 32t^3 - c(1+3t^2+3t^4+t^6) = 0.$

$\Rightarrow 6t+6t^5+12t^3-2at-2at^5-4at^3-b-bt^4-2bt^2 + bt^2+bt^6+2bt^4-32t^3-c-3ct^2-3ct^4-ct^6=0$

$\Rightarrow (b-c)t^6 - 2(a-3)t^5 + (b-3c)t^4 - 4(a+5)t^3 - (b+3c)t^2 - 2(a-3)t - (b+c) = 0 \longrightarrow (1)$

This is a 6th degree equation in t and hence has 6 roots $t_1, t_2, t_3, t_4, t_5, t_6$.

4(c) Prove that the equation $ah\sec\theta - bk\operatorname{cosec}\theta = a^2-b^2$ has four roots and that the sum of the values of θ which satisfy it, is equal to an odd multiple of π radians.

$\frac{ah}{\cos\theta} - \frac{bk}{\sin\theta} = a^2-b^2$

Solution: Given $ah\sec\theta - bk\operatorname{cosec}\theta = a^2-b^2 \rightarrow (1)$.

Let $t = \tan\theta/2$.

$\therefore (1) \Rightarrow ah\left(\frac{1+t^2}{1-t^2}\right) - bk\left(\frac{1+t^2}{2t}\right) = a^2-b^2$.

Since $\sin\theta = \frac{2t}{1+t^2}$, $\operatorname{cosec}\theta = \frac{1+t^2}{2t}$ & $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sec\theta = \frac{1+t^2}{1-t^2}$.

$\Rightarrow 2aht(1+t^2) - bk(1-t^2)(1+t^2) = 2t(a^2-b^2)(1-t^2)$

$\Rightarrow 2aht + 2aht^3 - bk(1-t^4) = 2t(a^2-b^2-a^2t^2+b^2t^2)$.

$\Rightarrow 2aht + 2aht^3 - bk + bkt^4 - 2ta^2 + 2tb^2 + 2a^2t^3 - 2b^2t^3 = 0$.

$\Rightarrow bkt^4 + 2(ah+a^2-b^2)t^3 + 2(ah-a^2+b^2)t - bk = 0 \rightarrow (2)$

(2) is a 4th degree equation in t and has 4 roots.

Let it be t_1, t_2, t_3, t_4.

Then $S_1 = \Sigma t_1 = \frac{-2(ah+a^2-b^2)}{bk}$

$S_2 = \Sigma t_1t_2 = 0$

$S_3 = \Sigma t_1t_2t_3 = -2\frac{(ah-a^2+b^2)}{bk}$; $S_4 = t_1t_2t_3t_4 = \frac{-bk}{bk} = -1$.

Now $\tan\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}\right) = \frac{S_1-S_3}{1-S_2+S_4}$

$$= \frac{-2\left(\frac{ah+a^2-b^2}{bk}\right) - 2\frac{(ah-a^2+b^2)}{bk}}{1-0+(-1)} = \infty.$$

$\Rightarrow \frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2} = n\pi + \pi/2$

$\Rightarrow \theta_1+\theta_2+\theta_3+\theta_4 = 2n\pi+\pi = (2n+1)\pi$.

$=$ an odd multiple of π.

$$S_1 = \Sigma t_1 = \frac{2(a-3)}{b-c}; \quad S_2 = \Sigma t_1 t_2 = \frac{b-3c}{b-c}.$$

$$S_3 = \Sigma t_1 t_2 t_3 = \frac{4(a+5)}{b-c}; \quad S_4 = \Sigma t_1 t_2 t_3 t_4 = -\frac{(b+3c)}{b-c}.$$

$$S_5 = \Sigma t_1 t_2 t_3 t_4 t_5 = \frac{2(a-3)}{b-c}; \quad S_6 = t_1 t_2 t_3 t_4 t_5 t_6 = -\frac{b+c}{b-c}.$$

Now

$$\tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6}{2}\right) = \frac{S_1 - S_3 + S_5}{1 - S_2 + S_4 - S_6} \longrightarrow (2)$$

$$1 - S_2 + S_4 - S_6 = 1 - \frac{(b-3c)}{b-c} - \frac{(b+3c)}{b-c} + \frac{(b+c)}{b-c}.$$

$$= \frac{\cancel{b}-\cancel{c}-\cancel{b}+\cancel{3c}-\cancel{b}-\cancel{3c}+\cancel{b}+\cancel{c}}{b-c} = 0.$$

$$\therefore (2) \Rightarrow \tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6}{2}\right) = \frac{S_1 - S_3 + S_5}{0} = \infty$$

since $S_1 - S_3 + S_5 \neq 0$.

$$\Rightarrow \frac{\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6}{2} = \frac{\pi}{2} + n\pi.$$

$$\Rightarrow \theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6 = 2n\pi + \pi$$

$$= (2n+1)\pi$$

= an odd multiple of π.

2_5

4(d) If $\tan\theta_1 + \tan\theta_2$, $\tan\theta_3$ $\tan\theta_4$ are the roots of ~~to~~ $x^4 + px^3 + qx^2 + rx + s = 0$, show that

$$\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{r-p}{1-q+s}.$$

Formations of equations.

Every n^{th} degree equation has n roots and only n roots.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0,$$

the relation between the roots and the coefficients are given by

$$\Sigma \alpha_1 = -\frac{a_1}{a_0}, \quad \Sigma \alpha_1 \alpha_2 = \frac{a_2}{a_0}, \ldots\ldots$$

$$\Sigma \alpha_1 \alpha_2 \alpha_3 = -\frac{a_3}{a_0}, \ldots$$

$$\alpha_1 \alpha_2 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

Using these results, we find the values of symmetric functions of some trigonometric ratios. That is, we use them in the formations of equations.

Formation of Equations.

(5) (a) Expand $\sin 7\theta$ as a polynomial in $\sin\theta$.
Hence obtain the cubic equation whose roots are
$\sin^2\frac{2\pi}{7}$, $\sin^2\frac{4\pi}{7}$, $\sin^2\frac{6\pi}{7}$.

Solution:

We know that $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta \longrightarrow (1)$

If $\theta = 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10}{7}, \frac{12\pi}{7}$, $\sin 7\theta = 0$.

Hence these 7 values of θ are the roots of the equation (1). $7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta = 0$.

Put $\sin\theta = x$.

$\Rightarrow 7x - 56x^3 + 112x^5 - 64x^7 = 0$ has roots $0, \sin\frac{2\pi}{7}, \sin\frac{4\pi}{7}, \sin\frac{6\pi}{7}, \sin\frac{8\pi}{7}, \sin\frac{10\pi}{7}, \sin\frac{12\pi}{7}$.

$\therefore 64x^6 - 112x^4 + 56x^2 - 7 = 0 \longrightarrow (2)$.

has roots $\sin\frac{2\pi}{7}, \sin\frac{4\pi}{7}, \sin\frac{6\pi}{7}, \sin\frac{8\pi}{7}, \sin\frac{10\pi}{7}, \sin\frac{12\pi}{7}$.

Now $\sin\frac{12\pi}{7} = \sin\left(2\pi - \frac{2\pi}{7}\right) = -\sin\frac{2\pi}{7}$.

$\sin\frac{10\pi}{7} = \sin\left(2\pi - \frac{4\pi}{7}\right) = -\sin\frac{4\pi}{7}$

$\sin\frac{8\pi}{7} = \sin\left(2\pi - \frac{6\pi}{7}\right) = -\sin\frac{6\pi}{7}$.

$\therefore$ (2) has roots $\pm\sin\frac{2\pi}{7}, \pm\sin\frac{4\pi}{7}, \pm\sin\frac{6\pi}{7}$.

Put $x^2 = y$ in (2). Then

$64y^3 - 112y^2 + 56y - 7 = 0$ has roots

$\sin^2\frac{2\pi}{7}, \sin^2\frac{4\pi}{7}, \sin^2\frac{6\pi}{7}$.

(5)(b) Find the equation whose roots are
$2\cos\frac{2\pi}{7}$, $2\cos\frac{4\pi}{7}$, $2\cos\frac{6\pi}{7}$.

Solution:

We know that

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$$

$$= 7 - 26(1-\cos 2\theta + 28(1-\cos 2\theta)^2 - 8(1-\cos 2\theta).$$

$$= 8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1.$$

When $\theta = \pm\pi/7, \pm\frac{2\pi}{7}, \pm\frac{3\pi}{7}$, $\sin 7\theta = 0$.

$\therefore$ $8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = 0$ has roots

$\pm\pi/7$, $\pm\frac{2\pi}{7}$, $\pm\frac{3\pi}{7}$.

Put $\boxed{\cos 2\theta = x}$

$\Rightarrow$ $8x^3 + 4x^2 - 4x - 1 = 0$ has roots $\cos(\pm 2\pi/7)$,

$\cos(\pm\frac{4\pi}{7})$, $\cos(\pm 6\pi/7)$.

(ie) $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, $\cos\frac{6\pi}{7}$.

Put $\boxed{2x = y}$

Then $y^3 + y^2 - 2y - 1 = 0$ has roots $2\cos\frac{2\pi}{7}$, $2\cos\frac{4\pi}{7}$, $2\cos\frac{6\pi}{7}$.

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(5)(c) Show that $\cos\frac{\pi}{9}\cdot\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{8}$.

Solution:

$\cos 9\theta = 256\cos^9\theta - 576\cos^7\theta + 432\cos^5\theta - 120\cos^3\theta + 9\cos\theta$. $\rightarrow$ (1)

When $\theta = 0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{12\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$, $\cos 9\theta = 1$.

Hence (1) has roots as the above nine values.

Put $\cos\theta = x$.

$\Rightarrow 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x - 1 = 0 \rightarrow$ (2)

has roots $1, \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \ldots \cos\frac{16\pi}{9}$.

Since 1 is a root of (2) $x-1$ is a factor.

Divide (2) by $x-1$, we get [explanation is given in page (19)]

$256x^8 + 256x^7 - 320x^6 - 320x^5 + 112x^4 + 112x^3 - 8x^2 - 8x + 1 = 0 \rightarrow$ (2)

Since $\cos\frac{16\pi}{9} = \cos\frac{2\pi}{9}$, $\cos\frac{14\pi}{9} = \cos\frac{4\pi}{9}$,

$\cos\frac{12\pi}{9} = \cos\frac{6\pi}{9}$, $\cos\frac{10\pi}{9} = \cos\frac{8\pi}{9}$.

The equation has repeated roots

$\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{6\pi}{9}, \cos\frac{8\pi}{9}$.

Finding the square root of (2), (explanation is given in page (20))

we have

$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0 \longrightarrow$ (3)

which has roots $\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{6\pi}{9}, \cos 8\pi/9$.

Now $\cos\frac{6\pi}{9} = \cos\frac{2\pi}{3} = -\frac{1}{2}$

$\Rightarrow (2x+1)$ is a factor of (3).

Dividing (3) by $(2x+1)$, we have

$$
\begin{array}{r}
8x^3 - 6x + 1 \\
2x+1 \,\overline{\big)\, 16x^4 + 8x^3 - 12x^2 - 4x + 1} \\
\underset{(-)}{16x^4} \underset{(-)}{+8x^3} \qquad\qquad\qquad \\
\hline
-12x^2 - 4x + 1 \\
\underset{(+)}{-12x^2} \underset{(+)}{-6x} \quad \\
\hline
2x + 1 \\
2x + 1 \\
\hline
0
\end{array}
$$

$8x^3 - 6x + 1 = 0$ has roots $\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$.

Product of roots $= \cos\frac{2\pi}{9} \cdot \cos\frac{4\pi}{9} \cdot \cos\frac{8\pi}{9} = -\frac{1}{8}$.

But $\cos 8\pi/9 = -\cos \pi/9$.

$$\therefore \boxed{\cos \pi/9 \; \cos\frac{2\pi}{9} \cdot \cos 4\pi/9 = 1/8}$$

(5) (d) Find the equation whose roots are $\tan \pi/5, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}$ and $\tan\frac{4\pi}{5}$.

Solution:

$$\tan 5\theta = \frac{5\tan\theta - 5C_3 \tan^3\theta + 5C_5 \tan^5\theta}{1 - 5C_2 \tan^2\theta + 5C_4 \tan^4\theta}$$

When $\theta = 0, \pi/5, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$, $\tan 5\theta = 0$

$\Rightarrow 5\tan\theta - 5C_3 \tan^3\theta + 5C_5 \tan^5\theta = 0 \longrightarrow (1)$

has roots $\tan\theta$, where $\theta = 0, \pi/5, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$.

Put $\tan\theta = x$, then (1) $\Rightarrow 5x - 10x^3 + x^5 = 0 \longrightarrow (2)$

Since 0 is a root of the equation, (2) becomes

$x^4 - 10x^2 + 5 = 0 \longrightarrow (3)$ has roots

$\tan \pi/5$, $\tan\frac{2\pi}{5}$, ~~and~~ $\tan\frac{3\pi}{5}$, $\tan\frac{4\pi}{5}$.

08/10/20 I year

Formation of Equations.

$256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x - 1 = 0$ [$(x-1)$ is a factor]

$\Rightarrow x = 1$ is a root

$$
\begin{array}{c|cccccccccc}
 & 256 & 0 & -576 & 0 & 432 & 0 & -120 & 0 & 9 & -1 \\
1 & 0 & 256 & 256 & -320 & -320 & 112 & 112 & \cancel{-08}\ -8 & -08 & 1 \\
\hline
 & 256 & 256 & -320 & -320 & 112 & 112 & \cancel{-08}\ -8 & -08 & 1 & \boxed{0}
\end{array}
$$

$$
\begin{array}{c|cccccccccc}
 & 256 & 0 & -576 & 0 & 432 & 0 & -120 & 0 & 9 & -1 \\
1 & 0 & 256 & 256 & -320 & -320 & 112 & 112 & -8 & -08 & 1 \\
\hline
 & 256 & 256 & -320 & -320 & 112 & 112 & -08 & -08 & 1 & \boxed{0}
\end{array}
$$

$\therefore$ The equation is

$256x^8 + 256x^7 - 320x^6 - 320x^5 + 112x^4 + 112x^3 - 8x^2 - 8x + 1 = 0$

$(2x+1)$ is a factor of $16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0$

$\Rightarrow x = -1/2$

$$
\begin{array}{c|ccccc}
 & 16 & 8 & -12 & -4 & 1 \\
-1/2 & 0 & -8 & 0 & +6 & -1 \\
\hline
 & 16 & 0 & -12 & +2 & \boxed{0}
\end{array}
$$

$16x^3 + 0x^2 - 12x + 2 = 0$

$\Rightarrow \boxed{8x^3 - 6x + 1 = 0}$

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To find the square root of

$$256x^8+256x^7-320x^6-320x^5+112x^4+112x^3-8x^2-8x+1=0.$$

$$
\begin{array}{r|l}
 & 16x^4+8x^3-12x^2-4x+1 \\
\hline
16x^4 & 256x^8+256x^7-320x^6-320x^5+112x^4+112x^3-8x^2-8x+1 \\
 & 256x^8 \\
\hline
32x^4+8x^3 & 256x^7-320x^6 \\
 & \underset{(-)}{256x^7}\underset{(-)}{+64x^6} \\
\hline
32x^4+16x^3-12x^2 & -384x^6-320x^5+112x^4 \\
 & \mp 384x^6 \mp 192x^5 \pm 144x^4 \\
\hline
32x^4+16x^3-24x^2-4x & -128x^5-32x^4+112x^3-8x^2 \\
 & \mp 128x^5 \mp 64x^4 \pm 96x^3 \pm 16x^2 \\
\hline
32x^4+16x^3-24x^2-8x+1 & 32x^4+16x^3-24x^2-8x+1 \\
 & 32x^4+16x^3-24x^2-8x+1 \\
\hline
 & 0
\end{array}
$$

∴ Square root is

$$\boxed{16x^4+8x^3-12x^2-4x+1=0.}$$

Sum of roots $= -\frac{8}{16} = -1/2.$

$x=-1/2 \Rightarrow 2x=-1 \Rightarrow (2x+1)$ is a factor

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5 (a) Prove that

$$\tan\frac{\pi}{11}\cdot\tan\frac{2\pi}{11}\cdot\tan\frac{3\pi}{11}\cdot\tan\frac{4\pi}{11}\cdot\tan\frac{5\pi}{11}=\sqrt{11}.$$

Proof: $\tan 11\theta = \dfrac{11\tan\theta - 11C_3\tan^3\theta + \cdots - \tan^{11}\theta}{1 - 11C_2\tan^2\theta + \cdots - 11\tan^{10}\theta}$.

Put $\boxed{\tan 11\theta = 0}$, we get

$11\tan\theta - 11C_3\tan^3\theta + \cdots - \tan^{11}\theta = 0. \rightarrow (1)$

has roots $\tan\theta$, where $\theta = 0, \frac{\pi}{11}, \frac{2\pi}{11}, \frac{3\pi}{11}, \cdots, \frac{10\pi}{11}$.

Put $\boxed{\tan\theta = x}$. Then (1) reduces to

$11x - 165x^3 + 462x^5 - 330x^7 + 55x^9 - x^{11} = 0 \rightarrow (2)$

Hence (2) has roots

$0, \tan\frac{\pi}{11}, \tan\frac{2\pi}{11}, \cdots \tan\frac{9\pi}{11}, \tan\frac{10\pi}{11}$.

Since $\tan\frac{10\pi}{11} = -\tan\frac{\pi}{11}$, $\tan\frac{9\pi}{11} = -\tan\frac{2\pi}{11}$,

$\tan\frac{8\pi}{11} = -\tan\frac{3\pi}{11}$, $\tan\frac{7\pi}{11} = -\tan 4\pi/11$,

$\tan\frac{6\pi}{11} = -\tan\frac{5\pi}{11}$.

$\therefore x^{10} - 55x^8 + 330x^6 - 462x^4 + 165x^2 - 11 = 0 \rightarrow (3)$

has roots $\pm\tan\frac{\pi}{11}, \pm\tan\frac{2\pi}{11}, \pm\tan\frac{3\pi}{11}, \pm\tan\frac{4\pi}{11}$,

$\pm\tan\frac{5\pi}{11}$.

Put $\boxed{x^2 = y}$. Then (3) $\Rightarrow$

$y^5 - 55y^4 + 330y^3 - 462y^2 + 165y - 11 = 0 \rightarrow (4)$

This equation has roots

$\tan^2\frac{\pi}{11}, \tan^2\frac{2\pi}{11}, \tan^2\frac{3\pi}{11}, \tan^2\frac{4\pi}{11}, \tan^2\frac{5\pi}{11}$.

$\therefore \tan^2\frac{\pi}{11}\cdot\tan^2\frac{2\pi}{11}\cdot\tan^2\frac{3\pi}{11}\cdot\tan^2\frac{4\pi}{11}\cdot\tan^2\frac{5\pi}{11} = 11.$

$$\Rightarrow \boxed{\tan\frac{\pi}{11}\cdot\tan\frac{2\pi}{11}\cdot\tan\frac{3\pi}{11}\cdot\tan\frac{4\pi}{11}\cdot\tan\frac{5\pi}{11} = \sqrt{11}}.$$

(5f) Expand $\tan 4\theta$ in terms of $\tan\theta$ and show that $\tan\frac{\pi}{16}$, $\tan\frac{5\pi}{16}$, $\tan\frac{9\pi}{16}$, $\tan\frac{13\pi}{16}$ are roots of the equation $x^4+4x^3-6x^2-4x+1=0$.

Solution:

$$\tan 4\theta = \frac{S_1 - S_3}{1-S_2+S_4} = \frac{4C_1\tan\theta - 4C_3\tan^3\theta}{1-4C_2\tan^2\theta+4C_4\tan^4\theta}$$

$$= \frac{4x-4x^3}{1-6x^2+x^4}, \text{ where } \boxed{x=\tan\theta}$$

Let θ denote any of the angles $\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$.

Then $4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$.

$= (n\pi + \pi/4)$, where $n = 0, 1, 2, 3$.

$\therefore \tan 4\theta = \tan(n\pi + \pi/4) = 1$.

$$\Rightarrow \frac{4x-4x^3}{1-6x^2+x^4} = 1.$$

$$x^4 - 6x^2+1 = 4x-4x^3.$$

$\Rightarrow x^4+4x^3-6x^2-4x+1=0$ has four roots $\tan\frac{\pi}{16}$, $\tan\frac{5\pi}{16}$, $\tan\frac{9\pi}{16}$, $\tan\frac{13\pi}{16}$.

(5g) Prove that $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\frac{4\pi}{7}\sin\frac{5\pi}{7}\sin\frac{6\pi}{7} = \frac{7}{64}$.

(or) $\sin^2\frac{\pi}{7}\cdot\sin^2\frac{2\pi}{7}\cdot\sin^2\frac{3\pi}{7} = \frac{7}{64}$.

Solution:

Consider $\theta = \frac{n\pi}{7}$.

$\Rightarrow \sin 7\theta = \sin n\pi = 0$.

Now $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^4\theta - 64\sin^7\theta = 0$.

$\therefore \sin\theta = 0$ (or)

$64\sin^6\theta - 112\sin^4\theta + 56\sin^2\theta - 7 = 0$

(ie) $64x^6 - 112x^4 + 56x^2 - 7 = 0 \rightarrow$ (2) where $\boxed{x = \sin\theta}$

$\sin\theta = 0 \Rightarrow n = 0$

The roots of (2) are

$\sin\frac{\pi}{7}, \sin\frac{2\pi}{7}, \sin\frac{3\pi}{7}, \sin\frac{4\pi}{7}, \sin\frac{5\pi}{7}, \sin\frac{6\pi}{7}$

But $\sin\frac{6\pi}{7} = \sin(\pi - \pi/7) = \sin\pi/7$.

$\sin\frac{5\pi}{7} = \sin\left(\pi - \frac{2\pi}{7}\right) = \sin\frac{2\pi}{7}$

$\sin\frac{4\pi}{7} = \sin\left(\pi - \frac{3\pi}{7}\right) = \sin\frac{3\pi}{7}$.

Product of roots $= \sin\frac{\pi}{7} \cdot \sin\frac{2\pi}{7} \cdot \sin\frac{3\pi}{7} \cdot \sin\frac{4\pi}{7} \cdot \sin\frac{5\pi}{7} \cdot \sin\frac{6\pi}{7} = \frac{7}{64}$

$$\Rightarrow \boxed{\sin^2\frac{\pi}{7} \cdot \sin^2\frac{2\pi}{7} \cdot \sin^2\frac{3\pi}{7} = \frac{7}{64}}$$

5h) Form the equation whose roots are $\sin^2\frac{2\pi}{7}, \sin^2\frac{4\pi}{7}, \sin^2\frac{6\pi}{7}$ and hence find the value of $\cos^2\frac{2\pi}{7} + \cos^2\frac{4\pi}{7} + \cos^2\frac{6\pi}{7}$.

<u>Solution:</u>

Consider $\theta = \frac{2n\pi}{7}, \; n = 0, 1, 2, \ldots, 6 \rightarrow$ (1)

Then $\sin 7\theta = \sin 2n\pi = 0 \Rightarrow 64\sin^7\theta - 112\sin^5\theta + 56\sin^3\theta - 7\sin\theta = 0$.

$\Rightarrow 64\sin^6\theta - 112\sin^4\theta + 56\sin^2\theta - 7 = 0, \; \sin\theta = 0$.

$\sin\theta = 0 \Rightarrow n = 0$.

$\therefore$ The roots of the equation (when $x = \sin\theta$)

$64x^6 - 112x^4 + 56x^2 - 7 = 0 \rightarrow$ (2) are

$\sin\frac{2\pi}{7}, \sin\frac{4\pi}{7}, \sin\frac{6\pi}{7}, \sin\frac{8\pi}{7}, \sin\frac{10\pi}{7}, \sin\frac{12\pi}{7}$.

But $\sin\frac{12\pi}{7} = \sin\left(2\pi - \frac{2\pi}{7}\right) = -\sin\frac{2\pi}{7}$.

$\sin\frac{10\pi}{7} = \sin\left(2\pi - \frac{4\pi}{7}\right) = -\sin\frac{4\pi}{7}$.

$\sin\frac{8\pi}{7} = \sin\left(2\pi - \frac{6\pi}{7}\right) = -\sin\frac{6\pi}{7}$.

The roots of (2) are $\pm\sin\frac{2\pi}{7}, \pm\sin\frac{4\pi}{7}, \pm\sin\frac{6\pi}{7}$.

Put $y = x^2$.

$\therefore$ The equation whose roots are

$\sin^2\frac{2\pi}{7}, \sin^2\frac{4\pi}{7}, \sin^2\frac{6\pi}{7}$ is

$64y^3 - 112y^2 + 56y - 7 = 0$.

Sum of the roots $= \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} + \sin^2\frac{6\pi}{7}$.

$= \frac{112}{64} = \frac{7}{4}$.

$\Rightarrow 1 - \cos^2\frac{2\pi}{7} + 1 - \cos^2\frac{4\pi}{7} + 1 - \cos^2\frac{6\pi}{7} = \frac{7}{4}$.

$\Rightarrow \boxed{\cos^2\frac{2\pi}{7} + \cos^2\frac{4\pi}{7} + \cos^2\frac{6\pi}{7} = \frac{5}{4}}$